SPATIAL GROWTH OF DISTURBANCES IN A COMPRESSIBLE BOUNDARY LAYER

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Excitation of unstable waves (Tollmien—Schlichting waves) due to the presence of one or the other type of free-stream disturbances with continuous spectra (acoustic, vortical, etc.) has been widely studied in recent times because of the need to predict boundary layer transition [1]. As a result of theoretical and experimental studies it has been established that the excitation of unstable waves takes place at spatial nonuniformities in the boundary layer flow [2]. The method of small disturbances which involves series solutions to linearized Navier—Stokes equations in terms of eigenfunctions for the locally homogeneous problem [3] is widely used to analytically investigate the Tollmien—Schlichting wave generation. Here the disturbances characterizing fixed frequency and spatial growth are analyzed. In view of this it becomes necessary to analyze the complete system of eigenfunctions of the linearized Navier—Stokes equations for the spatially growing disturbances, assuming locally homogeneous conditions. The growth of initial disturbances in incompressible boundary layer has been solved in [4] using Laplace transformation in time. The completeness of the system of eigenfunctions of the linearized Navier—Stokes equations describing the temporal growth of disturbances in incompressible boundary layer has been shown in [5] on the basis of results from [4]. For the spatially growing disturbances these equations remained unanswered. In the present paper an analysis has been carried out for the spatial growth of disturbances in compressible boundary layer.

1. Formulation of the Problem. Consider plane parallel boundary layer flow. Coordinates are chosen such that Ox is in the direction of the flow and the axis Oy is perpendicular to the plane surface. Let the length scale be the boundary layer thickness δ, let U_0 be the reference velocity, T_0 the reference temperature, ρ_0 the reference density, $\rho_0 U_0^2$ the reference pressure, μ_0 the reference viscosity, and δ/U_0 the reference time (the index 0 indicates quantities in the free stream, outside the boundary layer). The coefficient of viscosity is assumed to be a function of temperature. The vector function **A** is determined for the two-dimensional disturbances:

$$A_1 = u,\ A_2 = \partial u/\partial y,\ A_3 = v,\ A_4 = p',\ A_5 = \Theta,\ A_6 = \partial\Theta/\partial y,$$
$$A_7 = \partial u/\partial x,\ A_8 = \partial v/\partial x,\ A_9 = \partial\Theta/\partial x,$$

where u, v are the fluctuations in the x and y velocity components respectively; p' is the pressure fluctuation; Θ is the temperature fluctuation. We write the linearized Navier—Stokes equations after Fourier transformation in the form

$$\frac{\partial}{\partial y}\left(L_0\frac{\partial \mathbf{A}}{\partial y}\right) + L_1\frac{\partial \mathbf{A}}{\partial y} = H_1\mathbf{A} + H_2\frac{\partial \mathbf{A}}{\partial x}, \tag{1.1}$$

where L_0, L_1, H_1, and H_2 are 9×9 matrices. Their nonzero elements are given in the appendix. The initial and boundary conditions are formulated:

$$A_1 = A_3 = A_5 = 0 \ \text{at}\ \ y = 0,$$
$$|A_j| < \infty \ \ \text{at}\ \ y \to \infty\ (j = 1, \dots, 9), \tag{1.2}$$
$$\mathbf{A} = \mathbf{A}_0(y)\ \ \text{at}\ \ x = 0.$$

In formulating boundary conditions for the temperature fluctuation at y = 0 in (1.2), it was assumed that the solid surface is made of a highly conducting material [6].

According to Adamar [7], the mixed problem (1.1), (1.2) is incorrectly posed. However, it can be regularized by assuming that $\mathbf{A}_0(y)$ allows solutions to linearized Navier—Stokes equations $\mathbf{A}(x, y)$ with finite growth rate, i.e., there exist such positive constants M, s that $|A_j(x, y)| < Me^{sx} (j = 1, \dots, 9)$. Integral relations that the vector $\mathbf{A}_0$ should satisfy are obtained below in an explicit form.

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2. Solution to the Problem (1.1), (1.2). Using Laplace transforms to bring A(x, y) to the form $\mathbf{A}_p(p, y)$, where p is a complex variable, we get a system of equations

$$\frac{d}{dy}\left(L_0\frac{d\mathbf{A}_p}{dy}\right)+L_1\frac{d\mathbf{A}_p}{dy}=H_1\mathbf{A}_p+pH_2\mathbf{A}_p+\mathbf{F},\quad \mathbf{F}=-H_2\mathbf{A}_0. \tag{2.1}$$

Solution to the system (2.1) is sought by using variational method for the constant. In the homogeneous system (2.1), equations for the first six components are split and brought to the well known Lees—Lin type system [6] for the vector **z** describing the first six components of the vector $\mathbf{A}_p$:

$$d\mathbf{z}/dy=H_0\mathbf{z}, \tag{2.2}$$

where H_0 is a 6 ×6 matrix (its nonzero elements are given in the appendix). Here the components of A_{pj} (j =7, 8, 9) are uniquely determined from the first six components. The system (2.2) has six linearly independent solutions which have asymptotic dependence on y outside the boundary layer: $z_j \sim \exp(\lambda_j y)$ (j =1, ..., 6). The constants λ_j are determined by the relations [8]:

$$\lambda_{1,2}=\pm\sqrt{-p^2+\mathrm{Re}(p-i\omega)},$$

$$\lambda_{3,4}=\pm\left\{\frac{1}{2}(b_{22}+b_{33})+\sqrt{\frac{1}{4}(b_{22}-b_{33})^2+b_{23}b_{32}}\right\}^{1/2},$$

$$\lambda_{5,6}=\pm\left\{\frac{1}{2}(b_{22}+b_{33})-\sqrt{\frac{1}{4}(b_{22}-b_{33})^2+b_{23}b_{32}}\right\}^{1/2},$$

$$b_{11}=H_0^{21},\quad b_{22}=H_0^{42}H_0^{24}+H_0^{43}H_0^{34}+H_0^{46}H_0^{64},$$

$$b_{23}=H_0^{42}H_0^{25}+H_0^{43}H_0^{35}+H_0^{46}H_0^{65},\quad b_{32}=H_0^{64},\quad b_{33}=H_0^{65},$$

where H_0^{ij} are the elements of the matrix H_0 computed at $y\to\infty$; Re is the Reynolds number; ω is the disturbance frequency; i is the complex unit vector. We fix the branches Real $(\lambda_j) \sim 0$ (j =1, 3, 5) and to get a unique solution make a cut in the plane of the variable p according to the equations $\lambda_j^2=-k^2$ (j =1, 3, 5, k >0). For each linearly independent solution ξ_j (j =1, ..., 6) to the system (2.2) we write linearly independent solutions to the homogeneous system (2.1). The solution to (2.1) is sought in the form

$$\mathbf{A}_p=\boldsymbol{\xi}\mathbf{Q}(y)+\mathbf{Y},\ Y_i=0\ (i=1,\ldots,6),\ Y_i=-F_i\ (i=7,8,9),$$

where $\xi = ||\xi_1, \ldots, \xi_6||$ is the matrix of fundamental solutions (9 ×6); Q is the unknown vector-function with six components. For Q we get equations

$$2L_0\frac{d\xi}{dy}\frac{d\mathbf{Q}}{dy}+L_0\xi\frac{d^2\mathbf{Q}}{dy^2}+\frac{dL_0}{dy}\xi\frac{d\mathbf{Q}}{dy}+L_1\xi\frac{d\mathbf{Q}}{dy}=\mathbf{f},$$

$$f_1=F_1=0,\quad f_2=F_2-pH_2^{27}F_7+L_1^{28}\frac{dF_8}{dy},$$

$$f_3=F_3,\quad f_4=F_4-pH_2^{48}F_8,\quad f_5=F_5=0,$$

$$f_6=F_6-pH_2^{69}F_9,\quad f_j=0\quad (j=7,8,9). \tag{2.3}$$

The system (2.3) contains six equations to determine the six components of the vector Q. After simple transformations we bring (2.3) to the form

$$Zd\mathbf{Q}/dy=\boldsymbol{\varphi},\quad \varphi_1=0,\quad \varphi_2=F_2-pH_2^{27}+$$

$$+L_1^{28}dF_8/dy-L_1^{28}F_3,\quad \varphi_3=F_3,$$

$$\varphi_4=\left[-H_0^{33}F_3+F_4-pH_2^{48}F_8-\frac{d}{dy}\left(L_0^{43}F_3\right)\right]\Big/\left(1+L_0^{43}H_0^{34}\right),$$

$$\varphi_5=0,\quad \varphi_6=F_6-pH_2^{69}F_9, \tag{2.4}$$

where $Z = ||z_1, \ldots, z_6||$ is the matrix of fundamental solutions to the system (2.2). Solving (2.4) we get

$$\mathbf{A}_p=\sum_{j=1}^{6}\left(a_j+\int\limits_{y_j}^{y}\frac{D_j}{W}dy\right)\xi_j+\mathbf{Y},$$

$$W = \det Z,\ D_j/W = dQ_j/dy, \tag{2.5}$$

where the constants a_j, y_j should be obtained from boundary conditions at $y = 0$ and $y \to \infty$. Although the last three components in Eq. (2.5) do not express anything except their determination in the vector **A** as a derivative of other components with respect to x, their evaluation is essential because their value at $x = 0$ is fundamental to the solution. Further attention is concentrated on the first six components of the the vector $\mathbf{A}_p$. Hence we will drop the vector **Y**; the linearly independent solutions ξ_j are replaced by $\mathbf{z}_j$. In accordance with the choice of the branch λ_j and boundary conditions (1.2), $\mathbf{A}_p$ is expressed in the form

$$\mathbf{A}_p = \left(a_1 + \int_0^y \frac{D_1}{W}\,dy\right)\mathbf{z}_1 + \int_\infty^y \frac{D_2}{W}\,dy\,\mathbf{z}_2 + \left(a_3 + \int_0^y \frac{D_3}{W}\,dy\right)\mathbf{z}_3$$

$$+ \int_\infty^y \frac{D_4}{W}\,dy\,\mathbf{z}_4 + \left(a_5 + \int_0^y \frac{D_5}{W}\,dy\right)\mathbf{z}_5 + \int_\infty^y \frac{D_6}{W}\,dy\,\mathbf{z}_6,$$

$$a_1 = \left[\int_0^\infty \frac{D_2}{W}\,dyE_{235} + \int_0^\infty \frac{D_4}{W}\,dyE_{435} + \int_0^\infty \frac{D_6}{W}\,dyE_{635}\right]\Big/ E_{135},$$

$$a_3 = \left[\int_0^\infty \frac{D_2}{W}\,dyE_{125} + \int_0^\infty \frac{D_4}{W}\,dyE_{135} + \int_0^\infty \frac{D_6}{W}\,dyE_{165}\right]\Big/ E_{135},$$

$$a_5 = \left[\int_0^\infty \frac{D_2}{W}\,dyE_{132} + \int_0^\infty \frac{D_4}{W}\,dyE_{134} + \int_0^\infty \frac{D_6}{W}\,dyE_{136}\right]\Big/ E_{135},$$

$$E_{ijk} = \begin{Vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{3i} & z_{3j} & z_{3k} \\ z_{5i} & z_{5j} & z_{5k} \end{Vmatrix}, \tag{2.6}$$

where z_{ij} represents the i^{th} components of the j^{th} vector. Inverse Laplace transformation of (2.6) will be determined by the presence of branch points and poles. The function $W(y)$ from (2.5) can be found from consideration of the asymptotic expression for $\mathbf{z}_j$ as $y \to \infty$ using the well-known Jacobi function [9]:

$$W(y) = W(\infty)\exp\left[\int_\infty^y \mathrm{Sp}\,(H_0)\,dy\right].$$

The constant $W(\infty)$ cannot be zero because, when $W = 0$ for a certain value of p, the chosen system $\mathbf{z}_j$ becomes linearly dependent and it is then necessary to reconstruct the system of linearly independent vectors (similar to the analysis in [4]). The poles of (2.6) will be determined by the zeros of the function $E_{135}(p)$. It is well known [6] that the dispersion relation $E_{135} = 0$ determines the discrete spectrum of the linearized Navier—Stokes equations for Tollmien—Schlichting waves about which it is known, at least from the computed results, that all zeros of E_{135} have the real part Real(p) less than a certain finite number p_0. It could be as much greater than zero as it could be less than zero, depending on the parameters of the problem. Consider the structure of the cutouts in the plane of the complex variable p determined by the equations $\lambda_j^2 = -k^2$ ($j = 1, 3, 5, k > 0$). These equations are easily analyzed asymptotically as $k \to 0$ and $k \to \infty$. For finite values of the parameter k, Reynolds number, and the variable p, the equations were solved numerically. The structure of the cutouts thus obtained γ_j ($j = 1, \ldots, 7$) is shown in Fig. 1 (Mach number $M < 1$) and in Fig. 2 ($M > 1$). We note that one of the cutouts γ_7 for $k \to \infty$ has the limiting point p_* which it approaches along the line $\mathrm{Im}(p) = \mathrm{const}$. In the neighborhood of this point the solution to (2.6) is characterized by pressure fluctuation $\sim(p - p_*)^{-1/2}$. We also draw attention to the existence of three cuts found in the half-plane $\mathrm{Real}(p) > 0$ and extending to infinity. Their existence reflects the incorrectness of the problem (1.1), (1.2). If it is required from the initial condition $\mathbf{A}_0$ that the solution to (2.6) should be continuous across the cut for all p satisfying the inequality $\mathrm{Real}(p) > s$, then the problem will be regularized. Here, by closing the integration path Γ by an arc of a circle C_r going around all the points of the branch as shown in Fig. 2, we get as $r \to \infty$ the solution $A(x, y)$ as a result of inverse Laplace transform, as the sum of the residues of the discrete poles p_ν and integrals along two sides of the existing cuts.

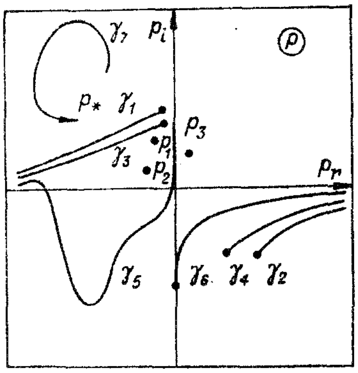

Fig. 1

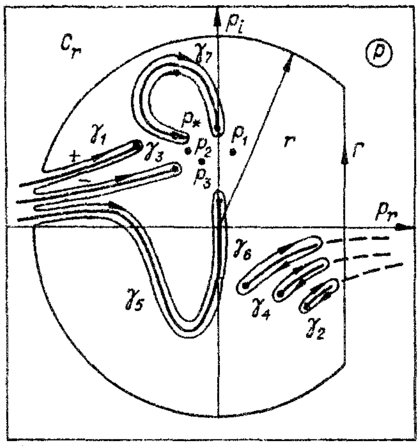

Fig. 2

Consider in detail the computation of the contribution of integrals along the two sides of the cut γ_1 to the inverse Laplace transform:

$$\mathbf{I}_1 = \frac{1}{2\pi i}\int_0^\infty \frac{dp}{dk} e^{px} (\mathbf{A}_p^+ - \mathbf{A}_p^-)\, dk,$$

where p(k) is determined by the equation $\lambda_1^2 = -k^2$, indices ± indicate the values of functions at the upper and lower branches of the cuts shown in Fig. 2. On the upper branch of the cut $\lambda_1 = -ik$, on the lower $\lambda_1 = ik$. Express all integrals in terms of the values of the functions at the upper branches of cuts. Here for γ_1 we have

$$W^- = -W^+,\quad D_1^- = -D_2^+,\quad D_2^- = -D_1^+,\quad D_j^- = -D_j^+$$
$$(j = 3, \ldots, 6),$$
$$E_{135}^- = E_{235}^+,\quad E_{235}^- = E_{135}^+,\quad E_{1jk}^- = E_{2jk}^+ \quad (j, k \neq 1, 2). \tag{2.7}$$

Using (2.7), after elementary but tedious transformations, we get

$$\mathbf{I}_1 = \frac{1}{2\pi i}\int_0^\infty e^{px}\frac{dp}{dk} G_1 [E_{235}\mathbf{z}_1 - E_{135}\mathbf{z}_2 + E_{125}\mathbf{z}_3 + E_{132}\mathbf{z}_5]_{\gamma_1}\, dk,$$

$$G_1 = \frac{1}{E_{135}E_{235}}\left[E_{135}\int_0^\infty \frac{D_1}{W}\,dy + E_{235}\int_0^\infty \frac{D_2}{W}\,dy - E_{345}\int_0^\infty \frac{D_4}{W}\,dy - E_{365}\int_0^\infty \frac{D_6}{W}\,dy\right]_{\gamma_1}, \tag{2.8}$$

where the index γ_1 denotes that p(k) is chosen in accordance with the equation for the cut γ_1. The contribution from the integrals along the branches of the cut γ_1 equals zero if $\mathbf{A}_p^+ = \mathbf{A}_p^-$. This will be satisfied when $G_1 = 0$. On the cut γ_2, also determined by the equation $\lambda_1^2 = -k^2$, we find that the difference $\mathbf{A}_p^+ - \mathbf{A}_p^-$ is proportional to the expression

$$G_2 = \frac{1}{E_{135}E_{235}}\left[E_{135}\int_0^\infty \frac{D_1}{W}\,dy + E_{235}\int_0^\infty \frac{D_2}{W}\,dy - E_{345}\int_0^\infty \frac{D_4}{W}\,dy - E_{365}\int_0^\infty \frac{D_6}{W}\,dy\right]_{\gamma_2}.$$

One of the conditions for regularizing the problem is $G_2 = 0$ when Real(p) > s. Similar conditions are obtained on the cuts γ_4, γ_6. Let us write the final result:

$$\mathbf{A}(x, y) = \Sigma' \operatorname{Res}(\mathbf{A}_p e^{px}) + \sum_{j=1}^{7} \mathbf{I}_j, \tag{2.9}$$

where $\mathbf{I}_j$ correspond to the contributions from the integrals along the branches of cuts γ_j (in view of the cumbersome expression their explicit form is not given); Σ' denotes summation of discrete set of poles. At present there are only numerical studies on discrete spectrum (analytical study is carried out only for the simplest cases [6]). Hence it is necessary to make the additional assumption that Σ' in (2.9) has significance.

3. Biorthogonal Vector System. The following biorthogonal system of vectors $\{\mathbf{A}_\alpha, \mathbf{B}_\alpha\}$ is determined:

$$\frac{d}{dy}\left(L_0\frac{d\mathbf{A}_\alpha}{dy}\right)+L_1\frac{d\mathbf{A}_\alpha}{dy}=H_1\mathbf{A}_\alpha+i\alpha H_2\mathbf{A}_\alpha, \tag{3.1}$$

$$A_{\alpha 1}=A_{\alpha 3}=A_{\alpha 5}=0 \text{ at } y=0, \ |A_{\alpha j}|<\infty \text{ as } y\to\infty \ (j=1,\ldots,9);$$

$$\frac{d}{dy}\left(L_0^*\frac{d\mathbf{B}_\alpha}{dy}\right)-L_1^*\frac{d\mathbf{B}_\alpha}{dy}=H_1^*\mathbf{B}_\alpha-i\bar{\alpha}H_2^*\mathbf{B}_\alpha,$$

$$B_{\alpha 2}=B_{\alpha 4}=B_{\alpha 6}=0 \text{ at } y=0, \ |B_{\alpha j}|<\infty \text{ as } y\to\infty \ (j=1,\ldots,9). \tag{3.2}$$

In Eqs. (3.1), (3.2), and in what follows α is a complex number; * denotes conjugate matrix, the bar above indicates complex conjugate. The system (3.1) coincides with the homogeneous problem (2.1). The system (3.2) determines the problem conjugate to (3.1).

Analysis of the spectrum of possible eigenvalues α for (3.1), (3.2) was initiated in [10]. There is a discrete spectrum corresponding to Tollmien—Schlichting waves and a continuous spectrum. Cuts γ_j in Figs. 1 and 2 actually correspond to the continuous spectrum of α if the equality $p=i\alpha$ is used. The following condition for orthogonality is valid:

$$\langle H_2\mathbf{A}_\alpha, \mathbf{B}_\beta\rangle=\Delta_{\alpha\beta}, \quad \langle\mathbf{A},\mathbf{B}\rangle=\int_0^\infty(\mathbf{A},\mathbf{B})\,dy\equiv\sum_{j=1}^{9}\int_0^\infty A_j\bar{B}_j dy, \tag{3.3}$$

where $\Delta_{\alpha\beta}=\delta_{\alpha\beta}$ is the Kronecker symbol, if one of the numbers belongs to the discrete spectrum; $\Delta_{\alpha\beta}=\delta(\alpha-\beta)$ is the delta function if both the numbers belong to the continuous spectrum [5].

The solution to (3.1) for the discrete spectrum can be written in the form

$$\mathbf{A}_\alpha=c_1\xi_1+c_3\xi_3+c_5\xi_5,$$

where one of the constants is arbitrary in view of the linearity of the problem and two others are determined from boundary conditions at $y=0$. Vectors ξ_j coincide with linearly independent solutions to the homogeneous system (2.1). Here and in what follows, results from earlier sections for $p=i\alpha$ are used. Since there are three boundary conditions at $y=0$, there is an eigenvalue problem $E_{135}(i\alpha)=0$ and a discrete spectrum results. Each of the branches of the continuous spectrum is obtained when for a certain $j\,(j=1,3,5)\,\lambda_j=\pm ik$ $(k>0)$. In this case, limited as $y\to\infty$, the solution is made up of four linearly independent solutions. Here all constants are determined accurately to the order of the choice of normalization of the solution to the problem (3.1). For example, for the continuous spectrum determined by the equation $\lambda_1=-ik$, the solution to (3.1) is written in the form

$$\mathbf{A}_\alpha=E_{235}\xi_1-E_{135}\xi_2+E_{125}\xi_3+E_{132}\xi_5,$$

which coincides with the vectorial part of the subintegral expression for I_1 in (2.8) and (2.9). Similarly, it is possible to show that all subintegral expressions of I_j in (2.9) are proportional to one of the eigensolutions to (3.1) from the continuous spectrum.

4. Completeness of the System of Eigenfunctions. Assuming the completeness of the system $\{\mathbf{A}_\alpha, \mathbf{B}_\alpha\}$, the formal solution to the problem (1.1) and (1.2) is written in the form

$$\mathbf{A}(x,y)=\Sigma'\langle H_2\mathbf{A}_0,\mathbf{B}_{\alpha\nu}\rangle e^{i\alpha_\nu x}\mathbf{A}_{\alpha\nu}+\sum_{j=1}^{7}\int_0^\infty\langle H_2\mathbf{A}_0,\mathbf{B}_{\alpha j}\rangle e^{i\alpha_j x}\mathbf{A}_{\alpha j}dk_j, \tag{4.1}$$

where α_ν corresponds to the solution of the discrete spectrum with the number ν; $\mathbf{A}_{\alpha\nu}$, $\mathbf{B}_{\alpha\nu}$ are solutions to (3.1) and (3.2), corresponding to α_ν, Σ' is the sum of all solutions of the discrete spectrum; α_j corresponds to the continuous spectrum with number j; $\mathbf{A}_{\alpha j}$, $\mathbf{B}_{\alpha j}$ are solutions to (3.1) and (3.2) when $\alpha=\alpha_j$; k_j is a real parameter that determines the continuous spectrum with number j. In order to regularize the problem (1.1) and (1.2) we require that from initial conditions $\mathbf{A}_0$ it should be possible to determine the solution with finite growth rate downstream. This requirement was already discussed while obtaining the solution with Laplace transform that in a formal representation of the solution (4.1) can also be expressed in the form of an integral relation $\langle H_2\mathbf{A}_0, \mathbf{B}_{\alpha j}\rangle=0$, which should be satisfied for all α_j with $\mathrm{Real}(i\alpha_j)>s$.

In order to establish the completeness of the system $\{\mathbf{A}_\alpha, \mathbf{B}_\alpha\}$ from (3.1) and (3.2) it is necessary to prove that the formal solution to (4.1) coincides with (2.9). It was made clear earlier that the subintegral expressions in I_j from (2.9) are proportional to $\exp(i\alpha_j)$ $\mathbf{A}_{\alpha j}$. On the basis of I_1 obtained by integrating along the cut γ_1, we show that G_1 from (2.8) is proportional to $\langle H_2\mathbf{A}_0, \mathbf{B}_{\alpha 1}\rangle$. G_1 is written in the form

$$G_1 = \frac{1}{E_{135}E_{235}} \int_0^\infty \left[E_{135}\frac{D_1}{W} + E_{235}\frac{D_2}{W} - E_{345}\frac{D_4}{W} - E_{365}\frac{D_6}{W} \right] dy.$$

Here D_j obtained from the solution of the algebraic system (2.4) are computed as the determinants of the matrix of fundamental solutions Z, in which the j-th column is replaced by the vector φ from (2.4). It is not difficult to show that $D_j/W = (\varphi, \chi_j)$, where χ_j is a linearly independent solution to the system of equations conjugate with (2.2) of the problem [11]:

$$-d\chi/dy = H_0^*\chi,$$
$$\chi_2 = \chi_4 = \chi_6 = 0 \text{ when } y = 0, \quad |\chi_j| < \infty \quad \text{when } y \to \infty \ (j = 1, \ldots, 6). \tag{4.2}$$

It is possible to verify that the vector function

$$\chi = E_{135}\chi_1 + E_{235}\chi_2 - E_{345}\chi_4 - E_{365}\chi_6$$

satisfies boundary conditions from (4.2). Then we find that G_1 is proportional to $\langle\varphi, \chi\rangle$. Direct computations show that there exists a correspondence between problems (3.2) and (4.2):

$$B_{\alpha 1} = \chi_1 - i\bar{\alpha}L_0^{43}\chi_4 d, \quad B_{\alpha 2} = \chi_2,$$
$$B_{\alpha 3} = \chi_3 d - i\bar{\alpha}\chi_2 d - L_0^{43}H_0^{33}\chi_4 - L_0^{43}\overline{H}_0^{64}\chi_6 d,$$
$$B_{\alpha 4} = \chi_4 d, \quad B_{\alpha 5} = \chi_5 + \overline{H}_0^{46}\chi_6, \quad B_{\alpha 6} = \chi_6,$$
$$d = (1 + L_0^{43}\overline{H}_0^{34}). \tag{4.3}$$

Using (4.3), solution φ from (2.4), we get $\langle\varphi, \chi_\alpha\rangle = \langle F, B_\alpha\rangle = -\langle H_2A_0, B_\alpha\rangle$, where the index α denotes that the solutions to (4.2) and (3.2) are found for the eigenvalue α. Thus, for all j = 1, ..., 7 it has been established that the subintegral expressions in I_j are proportional to $\langle H_2A_0, B_{\alpha j}\rangle$.

Proceeding along the lines used in [5], we come to the proportionality Σ' from (2.9) and (4.1). Further, in order to establish that all the final constants of proportionality are equal to one, we specify for A_0 one of the eigenfunctions A_α from (3.1). Using the orthogonality relation (3.3) and considering (2.9) at x = 0, the constants of proportionality are found to be equal to one. Thus we have proved that (2.9) is identical with (4.1) which establishes the completeness of the system $\{A_\alpha, B_\alpha\}$ from (3.1) and (3.2). We note that the proof could have been obtained without using the well-known properties of the solution to (2.9) which was obtained with Laplace transforms (at x = 0, it coincides with A_0). In order to do this it is necessary to compute the constant of proportionality for $\Delta_{\alpha\beta}$ in (3.3) for an arbitrary normalization of the solution to (3.1) and (3.2). This procedure was also followed but not discussed here because it is extremely tedious.

APPENDIX

$$L_0^{43} = -r\mu/\mathrm{Re}, \quad L_1^{ii} = 1 \quad (i = 1, \ldots, 6), \quad L_1^{28} = (m+1),$$
$$H_1^{21} = -i\omega\,\mathrm{Re}/\mu T, \quad H_1^{22} = -D\mu/\mu, \quad H_1^{23} = \mathrm{Re}\,DU/\mu T,$$
$$H_1^{25} = -D(\mu' DU)/\mu,$$
$$H_1^{26} = -\mu' DU/\mu, \quad H_1^{33} = DT/T, \quad H_1^{34} = i\omega\gamma M^2, \quad H_1^{35} = -i\omega/T, \quad H_1^{43} = $$
$$= i\omega/T,$$
$$H_1^{62} = -2(\gamma-1)M^2\sigma DU, \quad H_1^{63} = \mathrm{Re}\,\sigma DT/\mu T, \quad H_1^{64} = i\omega(\gamma-1)M^2\,\mathrm{Re}\,\sigma/\mu,$$
$$H_1^{65} = -D(\mu' DT)/\mu - i\omega\,\mathrm{Re}\,\sigma/\mu T, \quad H_1^{66} = -2D\mu/\mu,$$
$$H_2^{21} = \mathrm{Re}\,U/\mu T, \quad H_2^{23} = -D\mu/\mu, \quad H_2^{24} = \mathrm{Re}/\mu, \quad H_2^{27} = -r,$$
$$H_2^{31} = H_2^{69} = -1, \quad H_2^{34} = -\gamma M^2 U, \quad H_2^{35} = U/T, \quad H_2^{41} = mD\mu/\mathrm{Re},$$
$$H_2^{42} = (m+1)\mu/\mathrm{Re}, \quad H_2^{43} = -U/T, \quad H_2^{45} = \mu' DU/\mathrm{Re}, \quad H_2^{18} = \mu/\mathrm{Re},$$
$$H_2^{63} = -2(\gamma-1)M^2\sigma DU, \quad H_2^{64} = -(\gamma-1)M^2\,\mathrm{Re}\,\sigma U/\mu, \quad H_2^{65} = \mathrm{Re}\,U\sigma/\mu T,$$
$$H_0^{12} = H_0^{56} = 1, \quad H_0^{21} = -p^2 + (pU - i\omega)\,\mathrm{Re}/\mu T, \quad H_0^{22} = -D\mu/\mu,$$
$$H_0^{23} = -p(m+1)DT/T - pD\mu/\mu + \mathrm{Re}\,DU/\mu T,$$
$$H_0^{24} = p\,\mathrm{Re}/\mu + (m+1)\gamma M^2 p(pU - i\omega),$$

$$H_0^{25} = ip(m+1)(i\omega - pU)/T - D(\mu' DU)/\mu, \quad H_0^{26} = -\mu' DU/\mu, \quad H_0^{31} = -p,$$
$$H_0^{33} = DT/T, \quad H_0^{34} = \gamma M^2(i\omega - pU), \quad H_0^{35} = (pU - i\omega)/T,$$
$$H_0^{41} = -\beta p(rDT/T + 2D\mu/\mu), \quad H_0^{42} = -\beta p,$$
$$H_0^{43} = \beta[-p^2 + (i\omega - pU)\mathrm{Re}/\mu T + rD^2T/T + rD\mu DT/\mu T],$$
$$H_0^{44} = -\beta r\gamma M^2[pDU + (pU - i\omega)(DT/T + D\mu/\mu)],$$
$$H_0^{45} = \beta[rpDU/T + \mu' pDU/\mu - r(i\omega - pU)D\mu/\mu T],$$
$$H_0^{46} = -\beta r(i\omega - pU)/T, \quad H_0^{62} = -2(\gamma - 1)M^2\sigma DU,$$
$$H_0^{63} = -2(\gamma - 1)M^2\sigma pDU + \mathrm{Re}\,\sigma DT/\mu T, \quad H_0^{64} = (\gamma - 1)M^2\mathrm{Re}\,\sigma(i\omega - pU)/\mu,$$
$$H_0^{65} = -p^2 + \mathrm{Re}\,\sigma(pU - i\omega)/\mu T - (\gamma - 1)M^2\sigma\mu'(DU)^2/\mu - D^2\mu/\mu,$$
$$H_0^{66} = -2D\mu/\mu.$$

Here the following notation has been used: $U(y)$, $T(y)$ are velocity and temperature profiles of the mean flow; γ is the adiabatic index; M is the Mach number; Re is the Reynolds number; σ is Prandtl number; $r = (2/3)(e+2)$; $m = (2/3)(e - 1)$; e is the ratio of the second coefficient of viscosity to the first; $\mu(T)$ is the first coefficient of viscosity; $\mu' = d\mu/dT$; $D = d/dy$; $\beta = [\mathrm{Re}/\mu + r\gamma M^2(pU - i\omega)]^{-1}$; ω is the disturbance frequency.

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